

An explicit formula for obtaining $(q + 1, 8)$ -cages and others small regular graphs of girth 8

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Abstract

Let q be a prime power; $(q + 1, 8)$ -cages have been constructed as incidence graphs of a non-degenerate quadric surface in projective 4-space $P(4, q)$. The first contribution of this paper is a construction of these graphs in an alternative way by means of an explicit formula using graphical terminology. Furthermore by removing some specific perfect dominating sets from a $(q + 1, 8)$ -cage we derive k -regular graphs of girth 8 for $k = q - 1$ and $k = q$, having the smallest number of vertices known so far.

Keywords: Cages, girth, generalized quadrangles, perfect dominating sets.

1 Introduction

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow the book by Godsil and Royle [17] and the book by Lint and Wilson [22] for terminology and definitions.

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *girth* of a graph G is the number $g = g(G)$ of edges in a smallest cycle. For every $v \in V$, $N_G(v)$ denotes the

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neighbourhood of v , that is, the set of all vertices adjacent to v . The *degree* of a vertex $v \in V$ is the cardinality of $N_G(v)$. A graph is called *regular* if all the vertices have the same degree. A (k, g) -*graph* is a k -regular graph with girth g . Erdős and Sachs [12] proved the existence of (k, g) -graphs for all values of k and g provided that $k \geq 2$. Thus most work carried out has focused on constructing a smallest one [1, 2, 3, 4, 5, 7, 11, 13, 15, 16, 20, 21, 23, 24, 25]. A (k, g) -*cage* is a k -regular graph with girth g having the smallest possible number of vertices. Cages have been studied intensely since they were introduced by Tutte [26] in 1947. Counting the numbers of vertices in the distance partition with respect to a vertex yields a lower bound $n_0(k, g)$ with the precise form of the bound depending on whether g is even or odd:

$$n_0(k, g) = \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{(g-3)/2} & \text{if } g \text{ is odd;} \\ 2(1 + (k-1) + \cdots + (k-1)^{g/2-1}) & \text{if } g \text{ is even.} \end{cases} \quad (1)$$

Biggs [9] calls the *excess* of a (k, g) -graph G the difference $|V(G)| - n_0(k, g)$. The construction of graphs with small excess is a difficult task. Biggs is the author of a report on distinct methods for constructing cubic cages [10]. More details about constructions of cages can be found in the survey by Wong [27] or in the book by Holton and Sheehan [19] or in the more recent dynamic cage survey by Exoo and Jajcay [14].

A (k, g) -cage with $n_0(k, g)$ vertices and even girth exist only when $g \in \{4, 6, 8, 12\}$ [15]. If $g = 4$ they are the complete bipartite graph $K_{k,k}$, and for $g = 6, 8, 12$ these graphs are the incidence graphs of generalized $g/2$ -gons of order $k-1$. This is the main reason for (k, g) -cages with $n_0(k, g)$ vertices and even girth g are called *generalized polygon graphs* [9]. In particular a 3-gon of order $k-1$ is also known as a *projective plane* of order $k-1$. The 4-gons of order $k-1$ are called *generalized quadrangles* of order $k-1$, and, the 6-gons of order $k-1$, *generalized hexagons* of order $k-1$. All these objects are known to exist for all prime power values of $k-1$ [6, 17, 22], and no example is known when $k-1$ is not a prime power.

In this article we focus on the case $g = 8$. Let q be a prime power. Our main objective is to give an explicit construction of small $(k, 8)$ -graphs for $k = q-1, q$ and $q+1$. Next we present the contributions of this paper and in the following sections the corresponding proofs.

$(q+1, 8)$ -cages have been constructed by Benson [7] as follows. Let Q_4 be a non-degenerate quadric surface in projective 4-space $P(4, q)$. Define G_8 to be the graph whose vertices are the points and lines of Q_4 , two vertices being joined if and only if they correspond to an incident point-line pair in Q_4 . Then G_8 is a $(q+1)$ -regular graph of girth 8 with $n_0(q+1, 8)$ vertices. The first contribution of this paper is a construction of these graphs in an alternative way by means of an explicit formula given next.

Definition 1.1 Let \mathbb{F}_q be a finite field with $q \geq 2$ a prime power. Let $\Gamma_q = \Gamma_q[V_0, V_1]$ be a bipartite graph with vertex sets $V_r = \{(a, b, c)_r, (q, q, a)_r : a \in \mathbb{F}_q \cup \{q\}, b, c \in \mathbb{F}_q\}$, $r = 0, 1$, and

edge set defined as follows:

For all $a \in \mathbb{F}_q \cup \{q\}$ and for all $b, c \in \mathbb{F}_q$:

$$N_{\Gamma_q}((a, b, c)_1) = \begin{cases} \{(x, ax + b, a^2x + 2ab + c)_0 : x \in \mathbb{F}_q\} \cup \{(q, a, c)_0\} & \text{if } a \in \mathbb{F}_q; \\ \{(c, b, x)_0 : x \in \mathbb{F}_q\} \cup \{(q, q, c)_0\} & \text{if } a = q. \end{cases}$$

$$N_{\Gamma_q}((q, q, a)_1) = \{(q, a, x)_0 : x \in \mathbb{F}_q\} \cup \{(q, q, q)_0\}.$$

Theorem 1.1 *The graph Γ_q is a $(q + 1, 8)$ -cage on $2q^3 + 2q^2 + 2q + 2$ vertices for each prime power q .*

Remark 1.1 (i) *Let Γ_q be a $(q + 1, 8)$ -cage obtained in Theorem 1.1. Using geometrical terminology we call the elements of V_1 lines and the elements of V_0 points. Then Γ_q is the incidence graph of a classical generalized quadrangle $Q(4, q)$.*

(ii) *The edge set of a $(q + 1, 8)$ -cage Γ_q obtained in Theorem 1.1 can equivalently be expressed as follows:*

For all $x \in \mathbb{F}_q \cup \{q\}$ and for all $y, z \in \mathbb{F}_q$:

$$N_{\Gamma_q}((x, y, z)_0) = \begin{cases} \{(a, y - ax, a^2x - 2ay + z)_1 : a \in \mathbb{F}_q\} \cup \{(q, y, x)_1\} & \text{if } x \in \mathbb{F}_q; \\ \{(y, a, z)_1 : a \in \mathbb{F}_q\} \cup \{(q, q, y)_1\} & \text{if } x = q. \end{cases}$$

$$N_{\Gamma_q}((q, q, z)_0) = \{(q, a, z)_1 : a \in \mathbb{F}_q\} \cup \{(q, q, q)_1\};$$

$$N_{\Gamma_q}((q, q, q)_0) = \{(q, q, x)_1 : x \in \mathbb{F}_q \cup \{q\}\}.$$

Therefore, if q is even, $2a = 0$ for all $a \in \mathbb{F}_q$ yielding that if the partite sets V_0 and V_1 are interchanged the same graph Γ_q is obtained. Equivalently, if q is even (in geometrical terminology) the corresponding generalized quadrangle $Q(4, q)$ is said to be self-dual.

A bipartite graph is said to be *balanced* if each partite set has the same number of vertices. Let $q \geq 2$ be a prime power. In what follows we construct $(k, 8)$ -regular balanced bipartite graphs for $k = q - 1$ and $k = q$ with smallest known order. We will use the following notation. Given an integer $t \geq 1$, a graph G and a vertex $u \in V(G)$, let $N_G^t(u) = \{x \in V(G) : d_G(u, x) = t\}$, and $N_G^t[u] = \{x \in V(G) : d_G(u, x) \leq t\}$, where $d_G(u, x)$ denotes the distance between u and x in G . Note that $N_G^1(u) = N_G(u)$. A subset $U \subset V(G)$ is said to be a *perfect dominating set* of G if for each vertex $x \in V(G) \setminus U$, $|N_G(x) \cap U| = 1$ [18]. Let $\Gamma_q = \Gamma_q[V_0, V_1]$ be the $(q + 1, 8)$ -cage constructed in Theorem 1.1. Suppose that U is a perfect dominating set of Γ_q , then $\Gamma_q - U$ is a q -regular graph of girth 8. Thus it is of interest to find the largest perfect dominating set of Γ_q . In the following theorem we find perfect dominating sets of orders $2(q^2 + 1)$, $2(q^2 + 3q + 1)$ for any prime power q , and of order $2(q^2 + 4q + 3)$ for even prime powers q .

Theorem 1.2 *Let $q \geq 2$ be a prime power and $\Gamma_q = \Gamma_q[V_0, V_1]$ the $(q+1, 8)$ -cage constructed in Theorem 1.1. The following sets are perfect dominating in Γ_q :*

- (i) $A = N_{\Gamma_q}^2[\alpha] \cup N_{\Gamma_q}^2[\beta]$ where $\alpha, \beta \in V(\Gamma_q)$ and $\beta \in N_{\Gamma_q}^3(\alpha)$. Further $|A| = 2(q+1)^2$.
- (ii) $B = \bigcup_{c \in \mathbb{F}_q} N_{\Gamma_q}[(q, 0, c)_1] \cup N_{\Gamma_q}[(q, q, 0)_1] \cup \left(\bigcap_{c \in \mathbb{F}_q} N_{\Gamma_q}^2[(q, 0, c)_1] \cap N_{\Gamma_q}^2[(q, q, 0)_1] \right) \cup N_{\Gamma_q}^2[(q, q, \xi)_1]$,
where $\xi \in \mathbb{F}_q \setminus \{0\}$. Further $|B| = 2(q^2 + 3q + 1)$.
- (iii)

$$C = \bigcup_{x \in \mathbb{F}_q \cup \{q\}} N_{\Gamma_q}[(q, x, 0)_0] \cup \left(\bigcap_{x \in \mathbb{F}_q \cup \{q\}} N_{\Gamma_q}^2[(q, x, 0)_0] \right) \cup \bigcup_{x \in \mathbb{F}_q} N_{\Gamma_q}[(x, x, p(x))_1] \\ \cup N_{\Gamma_q}[(q, 1, 1)_1] \cup \left(\bigcap_{x \in \mathbb{F}_q} N_{\Gamma_q}^2[(x, x, p(x))_1] \cap N_{\Gamma_q}^2[(q, 1, 1)_1] \right),$$

where $q \geq 8$ is even and $p(x) = 1 + x + x^2$ for all $x \in \mathbb{F}_q$. Further $|C| = 2(q^2 + 4q + 3)$.

The perfect dominating sets described in item (ii) and (iii) of Theorem 1.2 are depicted in Figure 1 and in Figure 2 respectively.

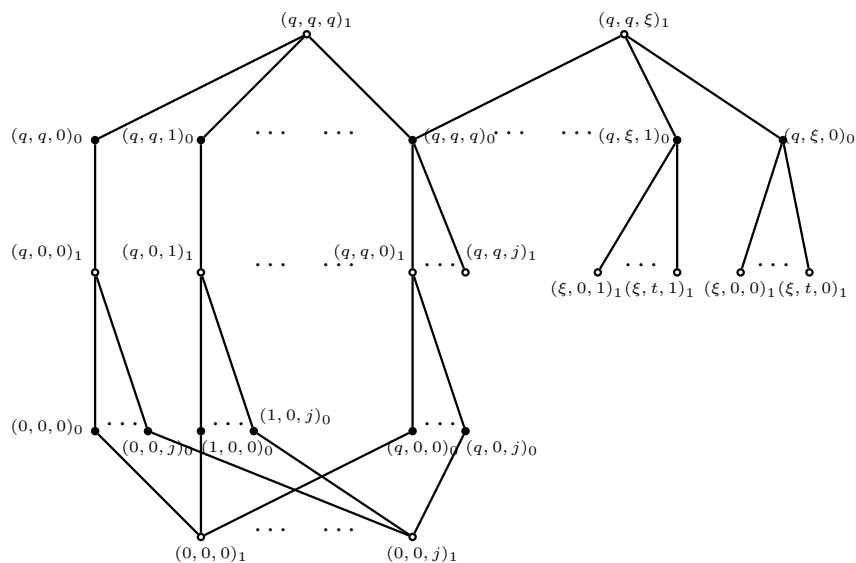


Figure 1: Deleted subgraph in (ii) of Theorem 1.3.

Remark 1.2 (a) Suppose $q = 2$. A cycle of length 8 is obtained by eliminating from the bipartite graph Γ_2 the vertices of the set B from Theorem 1.2 (ii). And the $(3, 8)$ -cage can be partitioned into the two induced subgraphs shown in Figure 2.

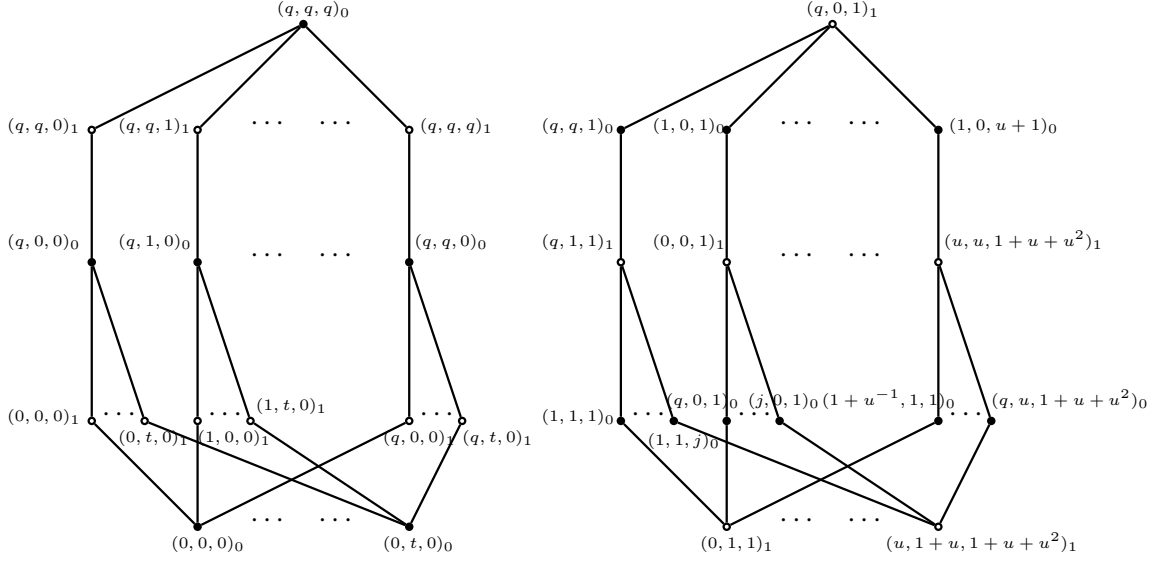


Figure 2: Deleted subgraph in (iii) of Theorem 1.3.

(b) For $q = 4$, $p(x) = 1 + x + x^2 \in \{0, 1\}$ for all $x \in \mathbb{F}_4$. Taking $\xi \in \mathbb{F}_4 \setminus \{0, 1\}$, we can find, for the $(5, 8)$ -cage, the following perfect dominating set similar to C of Theorem 1.2:

$$C' = \bigcup_{x \in \mathbb{F}_4} N_{\Gamma_4}[(4, x, \xi)_0] \cup N_{\Gamma_4}[(4, 4, 0)_0] \cup \left(\bigcap_{x \in \mathbb{F}_4 \cup \{4\}} N_{\Gamma_4}^2[(4, x, \xi)_0] \cap N_{\Gamma_4}[(4, 4, 0)_0] \right) \\ \bigcup_{x \in \mathbb{F}_4} N_{\Gamma_4}[(x, x, p(x))_1] \cup N_{\Gamma_4}[(4, 1, 1)_1] \cup \left(\bigcap_{x \in \mathbb{F}_4} N_{\Gamma_4}^2[(x, x, p(x))_1] \cap N_{\Gamma_4}^2[(4, 1, 1)_1] \right).$$

The following result is an immediate consequence of Theorem 1.2 and Remark 1.2 (b).

Theorem 1.3 Let $q \geq 2$ be a prime power and $\Gamma_q = \Gamma_q[V_0, V_1]$ the $(q+1, 8)$ -cage constructed in Theorem 1.1. Removing from Γ_q the perfect dominating sets from Theorem 1.2, q -regular graphs of girth 8 are obtained of orders $2q(q^2 - 1)$, $2q(q^2 - 2)$ for any prime power q or of order $2(q^3 - 3q - 2)$ for even prime powers $q \geq 4$.

Gács and Héger [16] obtain $(q, 8)$ -bipartite graphs on $2q(q^2 - 2)$ vertices if q is odd, or on $2(q^3 - 3q - 2)$ vertices if q is even, using a classical generalized quadrangle GQ and assuming that GQ has a substructure called regular point-pair (u, v) . Note that in Theorem 1.3 we obtain explicitly $(q, 8)$ -bipartite graphs on the same cardinality using Definition 1.1 without assuming anything. Moreover, using classical GQ , Beukemann and Metsch [8] prove that the cardinality of a perfect dominating set B is at most $|B| \leq 2(2q^2 + 2q)$ and if q is even $|B| \leq 2(2q^2 + q + 1)$. And $(k, 8)$ -regular balanced bipartite graphs for all prime powers q such that $3 \leq k \leq q$ of order $2k(q^2 - 1)$ have been obtained as subgraphs of the incidence graph of a generalized quadrangle [2]. This result has been improved by constructing $(k, 8)$ -regular balanced bipartite graphs of order $2q(kq - 1)$ in [5].

To finish we improve these results for the case $k = q - 1$.

Definition 1.2 Let $q \geq 4$ be a prime power and G_q the q -regular graph of girth 8 constructed in Theorem 1.3 on $2q(q^2 - 2)$ vertices choosing $\xi \in \mathbb{F}_q \setminus \{0, 1\}$.

Given a subset of vertices $S \subset V(G)$ we denote by $N_G(S) = \cup_{s \in S} N_G(s)$.

Theorem 1.4 Let $q \geq 4$ be a prime power and G_q the graph from Definition 1.2. Define $R = N_{G_q}(\{(q, y, z)_0 : y, z \in \mathbb{F}_q, y \neq 0, 1, \xi\}) \cap N_{G_q}^5((q, 1, 0)_0)$. The set

$$S := \bigcup_{z \in \mathbb{F}_q} N_{G_q}[(q, 1, z)_0] \cup N_{G_q}[R].$$

is perfect dominating in G_q . Furthermore, $G_q - S$ a $(q - 1)$ -regular graph of girth 8 of order $2q(q - 1)^2$.

2 $(q + 1, 8)$ -cages

In order to prove Theorem 1.1 we will first define two *auxiliary* graphs H_q and B_q (c.f. Definitions 2.1, 2.2, which were inspired by the construction of Lazebnik and Ustimenko [20] of a family of q -regular graphs $D(n, q)$, $n \geq 2$ and q a prime power, of order $2q^n$ and girth at least $n + 5$ for n odd (and at least $n + 4$ for n even). In particular when $n = 3$ the graph $D(3, q)$ has $2q^3$ vertices and girth 8. In what follows we construct another q -regular bipartite graph H_q of girth 8 as a first step to achieve our goal. It can be checked that $D(3, q)$ and H_q are not isomorphic for $q \geq 3$.

Definition 2.1 Let \mathbb{F}_q be a finite field with $q \geq 2$. Let $H_q = H_q[U_0, U_1]$ be a bipartite graph with vertex set $U_r = \mathbb{F}_q \times \mathbb{F}_q \times \mathbb{F}_q$, $r = 0, 1$; and edge set $E(H_q)$ defined as follows:

$$\text{For all } a, b, c \in \mathbb{F}_q : N_{H_q}((a, b, c)_1) = \{(x, ax + b, a^2x + c)_0 : x \in \mathbb{F}_q\}.$$

Lemma 2.1 Let H_q be the graph from Definition 2.1. For any given $a \in \mathbb{F}_q$, the vertices in the set $\{(a, b, c)_1 : b, c \in \mathbb{F}_q\}$ are mutually at distance at least four. Also, for any given $x \in \mathbb{F}_q$, the vertices in the set $\{(x, y, z)_0 : y, z \in \mathbb{F}_q\}$ are mutually at distance at least four.

Proof Suppose that there exists in H_q a path of length two $(a, b, c)_1(j, y, z)_0(a, b', c')_1$ with $b \neq b'$ or $c \neq c'$. Then $y = aj + b = aj + b'$ and $z = a^2j + c = a^2j + c'$. Hence $b = b'$ and $c = c'$ which is a contradiction. Similarly suppose that there exists a path $(x, y, z)_0(a, b, c)_1(x, y', z')_0$ with $y \neq y'$ or $z \neq z'$. Reasoning similarly, we obtain $y = ax + b = y'$, and $z = a^2x + c = z'$ which is a contradiction. ■

Proposition 2.1 *The graph H_q from Definition 2.1 is a q -regular bipartite of girth 8 and order $2q^3$.*

Proof For $q = 2$ it can be checked that H_2 consists of two disjoint cycles of length 8. Thus we assume that $q \geq 3$. Clearly H_q has order $2q^3$ and every vertex of U_1 has degree q . Let $(x, y, z)_0 \in U_0$. By definition of H_q ,

$$N_{H_q}((x, y, z)_0) = \{(a, y - ax, z - a^2x)_1 : a \in \mathbb{F}_q\}. \quad (2)$$

Hence every vertex of U_0 has also degree q and H_q is q -regular. Next, let us prove that H_q has no cycles of length less than 8. Otherwise suppose that there exists in H_q a cycle

$$C_{2t+2} = (a_0, b_0, c_0)_1(x_0, y_0, z_0)_0(a_1, b_1, c_1)_1 \cdots (x_t, y_t, z_t)_0(a_0, b_0, c_0)_1$$

of length $2t + 2$ with $t \in \{1, 2\}$. By Claim 0, $a_k \neq a_{k+1}$ and $x_k \neq x_{k+1}$ (subscripts being taken modulo $t + 1$). Then

$$\begin{aligned} y_k &= a_k x_k + b_k &= a_{k+1} x_k + b_{k+1}, & k = 0, \dots, t, \\ z_k &= a_k^2 x_k + c_k &= a_{k+1}^2 x_k + c_{k+1}, & k = 0, \dots, t, \end{aligned}$$

subscripts k being taken modulo $t + 1$. Summing all these equalities we get

$$\begin{aligned} \sum_{k=0}^{t-1} (a_k - a_{k+1}) x_k &= (a_1 - a_t) x_t, & t = 1, 2. \\ \sum_{k=0}^{t-1} (a_k^2 - a_{k+1}^2) x_k &= (a_1^2 - a_t^2) x_t, & t = 1, 2. \end{aligned} \quad (3)$$

If $t = 1$, then (3) leads to $(a_1 - a_1)(x_1 - x_0) = 0$. Then $a_1 = a_0$ or $x_1 = x_0$ which is a contradiction by Claim 0. This means that H_q has no squares so that we may assume that $t = 2$. The coefficient matrix of (3) has a Vandermonde determinant, i.e.:

$$\begin{vmatrix} a_1 - a_0 & a_0 - a_2 \\ a_1^2 - a_0^2 & a_0^2 - a_2^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_0 & a_2 \\ a_1^2 & a_0^2 & a_2^2 \end{vmatrix} = \prod_{0 \leq k < j \leq 2} (a_j - a_k)$$

This determinant is different from zero because by Claim 0, $a_{k+1} \neq a_k$ (the subscripts being taken modulo 3). Using Cramer's rule to solve it we obtain $x_1 = x_0 = x_2$ which is a contradiction with Claim 0.

Hence, H_q has girth at least 8. Furthermore, when $q \geq 3$ the minimum number of vertices of a q -regular bipartite graph of girth greater than 8 must be greater than $2q^3$. Thus we conclude that the girth of H_q is exactly 8. ■

Definition 2.2 *Let B_q be a bipartite graph with vertex set $V(B_q) = (\mathbb{F}_q^3, \mathbb{F}_q^3)$, and edge set $E(B_q)$ defined as follows:*

$$\text{For all } a, b, c \in \mathbb{F}_q : N_{B_q}((a, b, c)_1) = \{(j, aj + b, a^2j + 2ab + c)_0 : j \in \mathbb{F}_q\}.$$

Lemma 2.2 *The graph B_q is q -regular, has girth 8, order $2q^3$ and is isomorphic to the graphs H_q .*

Proof Let H_q be the bipartite graph from Definition 2.1. Since the map $\sigma : B_q \rightarrow H_q$ defined by $\sigma((a, b, c)_1) = (a, b, 2ab + c)_1$ and $\sigma((x, y, z)_0) = (x, y, z)_0$ is an isomorphism, the result holds. ■

Proof of Theorem 1.1: We will (re)-construct the graph Γ_q from the graph B_q adding some new vertices and edges. Reasoning as in Lemma 2.1 the following claim follows:

Claim 1: For any given $a \in \mathbb{F}_q$, the vertices of the set $\{(a, b, c)_1 : b, c \in \mathbb{F}_q\}$ are mutually at distance at least four in B_q . Also for any given $x \in \mathbb{F}_q$, the vertices of set $\{(x, y, z)_0 : y, z \in \mathbb{F}_q\}$ are mutually at distance at least four in B_q .

As a consequence of Claim 1 we obtain the following claim.

Claim 2: For all $x, y \in \mathbb{F}_q$, the q vertices of the set $\{(x, y, j)_0 : j \in \mathbb{F}_q\}$ are mutually at distance at least 6 in B_q .

Proof: By Claim 1, the q vertices $\{(x, y, j)_0 : j \in \mathbb{F}_q\}$ are mutually at distance at least 4. Suppose that B_q contains the following path of length four:

$$(x, y, j)_0 (a, b, c)_1 (x', y', j')_0 (a', b', c')_1 (x, y, j'')_0, \text{ for some } j'' \neq j.$$

Then $y = ax + b = a'x + b'$ and $y' = ax' + b = a'x' + b'$. It follows that $(a - a')(x - x') = 0$, which is a contradiction because $a \neq a'$ and $x \neq x'$ by Claim 1. □

Let $B'_q = B'_q[V_0, V'_1]$ be the bipartite graph obtained from $B_q = B_q[V_0, V_1]$ by adding q^2 new vertices to V_1 labeled $(q, b, c)_1$, $b, c \in \mathbb{F}_q$ (i.e., $V'_1 = V_1 \cup \{(q, b, c)_1 : b, c \in \mathbb{F}_q\}$), and new edges $N_{B'_q}((q, b, c)_1) = \{(c, b, j)_0 : j \in \mathbb{F}_q\}$ (see Figure 3). Then B'_q has $|V'_1| + |V_0| = 2q^3 + q^2$ vertices such that every vertex of V_0 has degree $q + 1$ and every vertex of V'_1 has still degree q . Note that the girth of B'_q is 8 by Claim 2. Further, Claim 1 partially holds in B'_q . We write this fact in the following claim.

Claim 3: For any given $a \in \mathbb{F}_q \cup \{q\}$, the vertices of the set $\{(a, b, c)_1 : b, c \in \mathbb{F}_q\}$ are mutually at distance at least four in B'_q .

Claim 4: For all $a \in \mathbb{F}_q \cup \{q\}$ and for all $c \in \mathbb{F}_q$, the q vertices of the set $\{(a, t, c)_1 : t \in \mathbb{F}_q\}$ are mutually at distance at least 6 in B'_q .

Proof: By Claim 3, for all $a \in \mathbb{F}_q \cup \{q\}$ the q vertices of $\{(a, t, c)_1 : t \in \mathbb{F}_q\}$ are mutually at distance at least 4 in B'_q . Suppose that there exists in B'_q the following path of length four:

$$(a, t, c)_1 (x, y, z)_0 (a', t', c')_1 (x', y', z')_0 (a, t'', c)_1, \text{ for some } t'' \neq t.$$

If $a = q$, then $x = x' = c$, $y = t$, $y' = t''$ and $a' \neq q$ by Claim 3. Then $y = a'x + t' = a'x' + t' = y'$ yielding that $t = t''$ which is a contradiction. Therefore $a \neq q$. If $a' = q$, then $x = x' = c'$ and

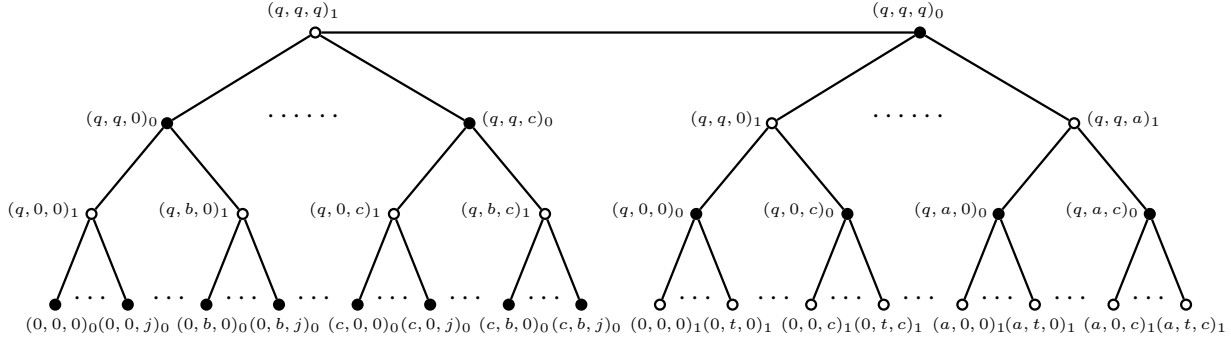


Figure 3: Spanning tree of Γ_q .

$y = y' = t'$. Thus $y = ax + t = ax' + t'' = y'$ yielding that $t = t''$ which is a contradiction. Hence we may assume that $a' \neq q$ and $a \neq a'$ by Claim 3. In this case we have:

$$\begin{aligned} y = ax + t &= a'x + t'; & z = a^2x + 2at + c &= a'^2x + 2a't' + c'; \\ y' = ax' + t'' &= a'x' + t'; & z' = a^2x' + 2at'' + c &= a'^2x' + 2a't' + c'. \end{aligned}$$

Hence

$$(a - a')(x - x') = t'' - t; \quad (4)$$

$$(a^2 - a'^2)(x - x') = 2a(t'' - t). \quad (5)$$

If q is even, (5) leads to $x = x'$ and (4) leads to $t'' = t$ which is a contradiction with our assumption. Thus assume q odd. If $a + a' = 0$, then (5) gives $2a(t'' - t) = 0$, so that $a = 0$ yielding that $a' = 0$ (because $a + a' = 0$) which is again a contradiction. If $a + a' \neq 0$, multiplying equation (4) by $a + a'$ and resting both equations we obtain $(2a - (a + a'))(t'' - t) = 0$. Then $a = a'$ because $t'' \neq t$, which is a contradiction to Claim 3. Therefore, Claim 4 holds. \square

Let $B''_q = B''_q[V'_0, V'_1]$ be the graph obtained from $B'_q = B'_q[V_0, V'_1]$ by adding $q^2 + q$ new vertices to V_0 labeled $(q, a, c)_0$, $a \in \mathbb{F}_q \cup \{q\}$, $c \in \mathbb{F}_q$, and new edges $N_{B''_q}((q, a, c)_0) = \{(a, t, c)_1 : t \in \mathbb{F}_q\}$ (see Figure 3). Then B''_q has $|V'_1| + |V'_0| = 2q^3 + 2q^2 + q$ vertices such that every vertex has degree $q + 1$ except the new added vertices which have degree q . Moreover the girth of B''_q is 8 by Claim 4.

Claim 5: For all $a \in \mathbb{F}_q \cup \{q\}$, the q vertices of the set $\{(q, a, j)_0 : j \in \mathbb{F}_q\}$ are mutually at distance at least 6 in B''_q .

Proof: Clearly these q vertices are mutually at distance at least 4 in B''_q . Suppose that there exists in B''_q the following path of length four:

$$(q, a, j)_0 (a, b, j)_1 (x, y, z)_0 (a, b', j')_1 (q, a, j')_0, \text{ for some } j' \neq j.$$

If $a = q$ then $x = j = j'$ which is a contradiction. Therefore $a \neq q$. In this case $y = ax + b = ax + b'$ which implies that $b = b'$. Hence $z = a^2x + 2ab + j = a^2x + 2ab' + j'$ yielding that $j = j'$ which is again a contradiction. \square

Let $B_q''' = B_q'''[V_0', V_1']$ be the graph obtained from B_q'' by adding $q + 1$ new vertices to V_1' labeled $(q, q, a)_1$, $a \in \mathbb{F}_q \cup \{q\}$, and new edges $N_{B_q'''}(q, q, a)_1 = \{(q, a, c)_0 : c \in \mathbb{F}_q\}$, see Figure 3. Then B_q''' has $|V_1'''| + |V_0'| = 2q^3 + 2q^2 + 2q + 1$ vertices such that every vertex has degree $q + 1$ except the new added vertices which have degree q . Moreover the girth of B_q''' is 8 by Claim 5 and clearly these $q + 1$ new vertices are mutually at distance 6. Finally, the $(q + 1, 8)$ -cage Γ_q is obtained by adding to B_q''' another new vertex labeled $(q, q, q)_0$ and edges $N_{\Gamma_q}((q, q, q)_0) = \{(q, q, i)_1 : i \in \mathbb{F}_q \cup \{q\}\}$. ■

2.1 Small $(q, 8)$ -graphs

Proof of Theorem 1.2: (i) Let $A = N_{\Gamma_q}^2[\alpha] \cup N_{\Gamma_q}^2[\beta]$ where $\alpha, \beta \in V(\Gamma_q)$ and $\beta \in N_{\Gamma_q}^3(\alpha)$. Since the girth of Γ_q is 8 there is a unique path of length three joining α and β . Hence $|N_{\Gamma_q}^2[\alpha] \cap N_{\Gamma_q}^2[\beta]| = |N_{\Gamma_q}(\alpha) \cap N_{\Gamma_q}^2(\beta)| + |N_{\Gamma_q}(\beta) \cap N_{\Gamma_q}^2(\alpha)| = 2$ yielding that $|A| = |N_{\Gamma_q}^2[\alpha] \cup N_{\Gamma_q}^2[\beta]| = 2(1 + q + 1 + (q + 1)q) - 2 = 2(q + 1)^2$.

Also since α and β are at distance three, $N_{\Gamma_q}^i(\alpha)$ and $N_{\Gamma_q}^i(\beta)$ are contained in different partite sets for all $i = 0, 1, 2, 3, 4$. Moreover, since the diameter of Γ_q is four, $V(\Gamma_q) = N_{\Gamma_q}^2[\alpha] \cup N_{\Gamma_q}^3(\alpha) \cup N_{\Gamma_q}^4(\alpha) = N_{\Gamma_q}^2[\beta] \cup N_{\Gamma_q}^3(\beta) \cup N_{\Gamma_q}^4(\beta)$. Hence if $v \notin A$ then $v \in N_{\Gamma_q}^3(\alpha) \cup N_{\Gamma_q}^4(\alpha)$. If $v \in N_{\Gamma_q}^3(\alpha)$ then $|N_{\Gamma_q}(v) \cap A| = |N_{\Gamma_q}(v) \cap N_{\Gamma_q}^2(\alpha)| = |N_{\Gamma_q}(v) \cap N_{\Gamma_q}^2[\alpha]| = 1$ because the girth is 8. If $v \in N_{\Gamma_q}^4(\alpha)$ then $|N_{\Gamma_q}(v) \cap A| = |N_{\Gamma_q}(v) \cap N_{\Gamma_q}^2(\beta)| = 1$. Therefore A is a perfect dominating set of Γ_q .

(ii) From Theorem 1.1, it follows that $\bigcap_{c \in \mathbb{F}_q} N_{\Gamma_q}^2[(q, 0, c)_1] \cap N_{\Gamma_q}^2[(q, q, 0)_1] = \{(q, q, q)_1\} \cup \{(0, 0, c)_1 : c \in \mathbb{F}_q\}$ and $N_{\Gamma_q}^2[(q, q, \xi)_1] = \bigcup_{j \in \mathbb{F}_q} N_{\Gamma_q}[(q, \xi, j)_0] \cup N_{\Gamma_q}[(q, q, q)_0]$. Let us denote by $F = \bigcup_{c \in \mathbb{F}_q} N_{\Gamma_q}[(q, 0, c)_1] \cup N_{\Gamma_q}[(q, q, 0)_1] \cup \{(0, 0, c)_1 : c \in \mathbb{F}_q\}$. We can check that $F \cap N_{\Gamma_q}^2[(q, q, \xi)_1] = \{(q, q, q)_0, (q, q, 0)_1\}$ (see Figure 1). Hence $|B| = |N_{\Gamma_q}^2[(q, q, \xi)_1]| + |F| - 2 = 1 + (q + 1) + q(q + 1) + (q + 1)(q + 2) + q - 2 = 2q^2 + 6q + 2$. Let us prove that B is a perfect dominating set.

For all vertices $(x, y, z)_0 \in V_0 \setminus B$ with $x \in \mathbb{F}_q \cup \{q\}$, $y, z \in \mathbb{F}_q$ we have:

$$\begin{aligned} N_{\Gamma_q}((x, y, z)_0) \cap B &= N_{\Gamma_q}((x, y, z)_0) \cap N_{\Gamma_q}^2[(q, q, \xi)_1] \\ &= \begin{cases} \{(\xi, y - \xi x, \xi^2 x - 2\xi y + z)_1\} \subset N_{\Gamma_q}[(q, \xi, y)_0] & \text{if } x \neq q; \\ \{(q, q, y)_1\} \subset N_{\Gamma_q}[(q, q, q)_0] & \text{if } x = q. \end{cases} \end{aligned}$$

Moreover, observe that $N_{\Gamma_q}((q, 0, c)_1) \setminus \{(q, q, c)_0\} = \{(c, 0, j)_0 : j \in \mathbb{F}_q\}$; and $N_{\Gamma_q}((0, 0, c)_1) = \{(x, 0, c)_0 : x \in \mathbb{F}_q \cup \{q\}\}$, see Figure 1. Then

$$F \cap V_0 = \{(x, 0, c)_0 : x \in \mathbb{F}_q \cup \{q\}, c \in \mathbb{F}_q\} \cup \{(q, q, x)_0 : x \in \mathbb{F}_q \cup \{q\}\}.$$

Also, for all vertices $(a, b, c)_1 \in V_1 \setminus B$ with $a \in \mathbb{F}_q \cup \{q\}$, $b, c \in \mathbb{F}_q$ we have:

$$N_{\Gamma_q}((a, b, c)_1) \cap B = N_{\Gamma_q}((a, b, c)_1) \cap F = \begin{cases} \{(-a^{-1}b, 0, ab + c)_0\} & \text{if } a \neq 0, q; \\ \{(q, 0, c)_0\} & \text{if } a = 0; \\ \{(q, q, c)_0\} & \text{if } a = q. \end{cases}$$

Therefore B is a perfect dominating set of Γ_q .

(iii) Let denote $R_0 = \bigcup_{x \in \mathbb{F}_q \cup \{q\}} N_{\Gamma_q}[(q, x, 0)_0] \cup \left(\bigcap_{x \in \mathbb{F}_q \cup \{q\}} N_{\Gamma_q}^2[(q, x, 0)_0] \right)$. Theorem 1.1, yields that $N_{\Gamma_q}((q, x, 0)_0) \setminus \{(q, q, x)_1\} = \{(x, a, 0)_1 : a \in \mathbb{F}_q\}$; and $N_{\Gamma_q}((0, y, 0)_0) = \{(a, y, -2ay)_1 : a \in \mathbb{F}_q\} \cup \{(q, y, 0)_1\}$. Since q is even, $-2ay = 0$ and therefore

$$\bigcup_{x \in \mathbb{F}_q \cup \{q\}} N_{\Gamma_q}((q, x, 0)_0) = \bigcup_{y \in \mathbb{F}_q} N_{\Gamma_q}((0, y, 0)_0) \cup N_{\Gamma_q}((q, q, q)_0) \text{ (see Figure 2).}$$

Hence $\bigcap_{x \in \mathbb{F}_q \cup \{q\}} N_{\Gamma_q}^2[(q, x, 0)_0] = \{(0, y, 0)_0 : y \in \mathbb{F}_q\} \cup \{(q, q, q)_0\}$, implying that $|R_0| = (q+1)^2 + 2(q+1)$.

$$\text{Let } R_1 = \bigcup_{x \in \mathbb{F}_q} N_{\Gamma_q}[(x, x, p(x))_1] \cup N_{\Gamma_q}[(q, 1, 1)_1] \cup \left(\bigcap_{x \in \mathbb{F}_q} N_{\Gamma_q}^2[(x, x, p(x))_1] \cap N_{\Gamma_q}^2[(q, 1, 1)_1] \right).$$

By Theorem 1.1, it is not difficult to check that $\{(x, x, p(x))_1 : x \in \mathbb{F}_q\} \cup \{(q, 1, 1)_1\}$ is a set of $q+1$ vertices mutually at distance four in Γ_q . Also $\{(x, 1+x, p(x))_1 : x \in \mathbb{F}_q\} \cup \{(q, 0, 1)_1\}$ is a set of $q+1$ vertices mutually at distance four in Γ_q . Let us show that

$$\bigcup_{x \in \mathbb{F}_q} N_{\Gamma_q}((x, x, p(x))_1) \cup N_{\Gamma_q}((q, 1, 1)_1) = \bigcup_{x \in \mathbb{F}_q} N_{\Gamma_q}((x, 1+x, p(x))_1) \cup N_{\Gamma_q}((q, 0, 1)_1). \quad (6)$$

Note that the sets on both sides have the same cardinality, then to prove the equality it is enough to show one inclusion. We have

$$\begin{aligned} N_{\Gamma_q}((x, x, p(x))_1) &= \{(j, xj + x, x^2j + p(x))_0 : j \in \mathbb{F}_q\} \cup \{(q, x, p(x))_0\}, \text{ for all } x \in \mathbb{F}_q; \\ N_{\Gamma_q}((q, 1, 1)_1) &= \{(1, 1, j)_0 : j \in \mathbb{F}_q\} \cup \{(q, q, 1)_0\}. \end{aligned}$$

Furthermore, since q is even,

$$\begin{aligned} N_{\Gamma_q}((x, 1+x, p(x))_1) &= \{(j, jx + 1 + x, x^2j + p(x))_0 : j \in \mathbb{F}_q\} \cup \{(q, x, p(x))_0\}, \text{ for all } x \in \mathbb{F}_q; \\ N_{\Gamma_q}((q, 0, 1)_1) &= \{(1, 0, j)_0 : j \in \mathbb{F}_q\} \cup \{(q, q, 1)_0\}. \end{aligned}$$

We can check that

$$\begin{aligned} N_{\Gamma_q}((x, x, p(x))_1) \cap N_{\Gamma_q}((x, 1+x, p(x))_1) &= \{(q, x, p(x))_0\}, \text{ for all } x \in \mathbb{F}_q; \\ N_{\Gamma_q}((q, 1, 1)_1) \cap N_{\Gamma_q}((q, 0, 1)_1) &= \{(q, q, 1)_0\}. \end{aligned}$$

For all $j \in \mathbb{F}_q$, $j \neq 1$, $(j, xj + x, x^2j + p(x))_0 \in N_{\Gamma_q}((x, x, p(x))_1) \cap N_{\Gamma_q}((x, 1+x, p(x))_1)$ where $v = (1+j)^{-1} + x$ because q is even. And $(1, 0, x+1)_0 \in N_{\Gamma_q}((x, x, p(x))_1) \cap N_{\Gamma_q}((q, 0, 1)_1)$ (see

Figure 2) because $p(x) = 1 + x + x^2$ and $p(x) + x^2 = 1 + x$. Furthermore, for all $j \in \mathbb{F}_q$, $(1, 1, j)_0 \in N_{\Gamma_q}((q, 1, 1)_1) \cap N_{\Gamma_q}((a, 1 + a, 1 + a + a^2)_1)$ where $a = 1 + j$. Hence equality (6) holds. This implies that $\bigcap_{x \in \mathbb{F}_q} N_{\Gamma_q}^2[(x, x, p(x))_1] \cap N_{\Gamma_q}^2[(q, 1, 1)_1] = \{(x, 1 + x, p(x))_1 : x \in \mathbb{F}_q\} \cup \{(q, 0, 1)_1\}$. Thus $|R_1| = (q + 1)^2 + 2(q + 1)$.

To finish the proof note that every vertex $f \in R_r$, $r = 0, 1$, with $|N_{\Gamma_q}(f) \cap R_r| = 2$ has exactly one neighbor in R_{r+1} and $q - 2$ more neighbors in $V(\Gamma_q) \setminus C$. Moreover, every vertex $v \in V(\Gamma_q) \setminus C$ has $|N_{\Gamma_q}(v) \cap C| \leq 1$ because the diameter of the subgraph induced by $C = R_0 \cup R_1$ is 5 and the girth of Γ_q is 8. This implies that $|N_{G_q}(R_0 \cup R_1) \cap V(\Gamma_q) \setminus C| = 2(q - 2)(q + 1)^2 = 2(q^3 - 3q - 2) = |V(\Gamma_q) \setminus C|$ yielding that $|N_{\Gamma_q}(v) \cap C| = 1$ for all $v \in V(\Gamma_q) \setminus C$. Therefore C is a perfect dominating set. ■

Lemma 2.3 *Let G_q be the graph from Definition 1.2. Define $P = \{(q, y, z)_0 : y, z \in \mathbb{F}_q, y \neq 0, 1, \xi\}$ and $R = N_{G_q}(P) \cap N_{G_q}^5((q, 1, 0)_0)$. Then $|R| = |P| = q(q - 3)$, $|N_{G_q}(R)| = 2q(q - 2)$ and every $v \in N_{G_q}(R) \setminus P$ has exactly 1 neighbor in $N_{G_q}^5((q, 1, 0)_0) \setminus R$.*

Proof First, note that for all $y \in \mathbb{F}_q \setminus \{0, \xi\}$ the set of q vertices $\{(q, y, z)_0 : z \in \mathbb{F}_q\}$ are mutually at distance 6 in G_q because they were q neighbors in Γ_q of the removed vertex $(q, q, y)_1$. Moreover, the vertices $(x, 0, z)_0$ with second coordinate zero have been removed from Γ_q to obtain G_q . Therefore according to Definition 1.1, the paths of length four in G_q joining $(q, 1, 0)_0$ and a vertex from P are as follows (see Figure 4):

$$(q, 1, 0)_0 (1, b, 0)_1 (x, x + b, x + 2b)_0 (y, t, z)_1 (q, y, z)_0, \text{ for all } b, x, t \in \mathbb{F}_q \text{ such that } b + x \neq 0.$$

Hence $x + b = xy + t$ and $x + 2b = y^2x + 2yt + z$ for all $b, x, t \in \mathbb{F}_q$ such that $b + x \neq 0$. The claim follows because if $x + b = xy + t = 0$, then $x + 2b = y^2x + 2yt + z$ gives that $t = (1 - y^2)^{-1}yz$, that is, $(y, (1 - y^2)^{-1}yz, z)_1 \in R$ is the unique neighbor in R of $(q, y, z)_0 \in P$. Therefore every $(q, y, z)_0 \in P$ has a unique neighbor $(y, b, z)_1 \in R$ yielding that $|R| = |P| = q(q - 3)$.

It follows that every $v \in N_{G_q}(R) \setminus P$ has at most $|R|/q = q - 3$ neighbors in R because for each y the vertices from the set $\{(q, y, z)_0 : z \in \mathbb{F}_q\} \subset P$ are mutually at distance 6. Furthermore, every $v \in N_{G_q}(R) \setminus P$ has at most one neighbor in $N_{G_q}^5((q, 1, 0)_0) \setminus R$ because the vertices $\{(q, 1, z)_0 : z \in \mathbb{F}_q, z \neq 0\}$ are mutually at distance 6. Therefore every $v \in N_{G_q}(R) \setminus P$ has at least two neighbors in $N_{G_q}^3((q, 1, 0)_0)$. Thus denoting $K = N_{G_q}(N_{G_q}(R) \setminus P) \cap N_{G_q}^3((q, 1, 0)_0)$ we have

$$|K| \geq 2|N_{G_q}(R) \setminus P|. \quad (7)$$

Moreover, observe that $(N_{G_q}(P) \setminus R) \cap K = \emptyset$ and since the elements of P are mutually at distance at least 4 we obtain that $|N_{G_q}(P) \setminus R| = q|P| - |R| = (q - 1)|P|$. Hence

$$|N_{G_q}^3((q, 1, 0)_0)| \geq |N_{G_q}(P) \setminus R| + |K| = (q - 1)|P| + |K|.$$

Since $|N_{G_q}^3((q, 1, 0)_0)| = q(q - 1)^2$ and $|P| = q(q - 3)$ we obtain that $|K| \leq 2q(q - 1)$ yielding by (7) that $|N_{G_q}(R) \setminus P| \leq q(q - 1)$. As P contains at least q elements mutually at distance 6,

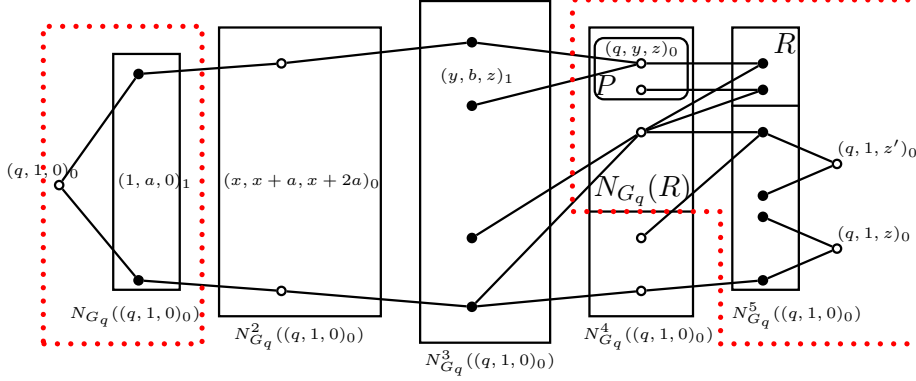


Figure 4: Structure of the graph G_q . The eliminated vertices are inside the dotted box.

so R contains at least q elements mutually at distance 4, thus we have $|N_{G_q}(R) \setminus P| \geq q^2 - q$. Therefore $|N_{G_q}(R) \setminus P| = q^2 - q$ and all the above inequalities are actually equalities. Thus $|N_{G_q}(R)| = q^2 - q + |P| = 2q(q - 2)$ and every $v \in N_{G_q}(R) \setminus P$ has exactly 1 neighbor in $N_{G_q}^5((q, 1, 0)_0) \setminus R$. ■

Proof of Theorem 1.4: Let G_q be the q -regular graph from Definition 1.2 and consider the sets $P = \{(q, y, z)_0 : y, z \in \mathbb{F}_q, y \neq 0, 1, \xi\}$ and $R = N_{G_q}(P) \cap N_{G_q}^5((q, 1, 0)_0)$. In particular the vertices of the set $\{(q, 1, z)_0 : z \in \mathbb{F}_q\}$ are mutually at distance 6, then by Lemma 2.3, we have

$$\begin{aligned}
 |N_{G_q}^4((q, 1, 0)_0) \setminus N_{G_q}(R)| &= \left| \bigcup_{z \in \mathbb{F}_q \setminus \{0\}} (N_{G_q}^2((q, 1, z)_0) \cup P) \setminus N_{G_q}(R) \right| \\
 &= q(q-1)^2 + q(q-3) - 2q(q-2) \\
 &= q(q-1)(q-2).
 \end{aligned}$$

Let us denote by $E[A, B]$ the set of edges between two set of vertices A and B . Then $|E[N_{G_q}^3((q, 1, 0)_0), N_{G_q}^4((q, 1, 0)_0)]| = q(q-1)^3$ and $|E[N_{G_q}^3((q, 1, 0)_0), N_{G_q}^4((q, 1, 0)_0) \setminus N_{G_q}(R)]| = q(q-1)^2(q-2)$. Therefore, $|E[N_{G_q}^3((q, 1, 0)_0), N_{G_q}(R)]| = q(q-1)^3 - q(q-1)^2(q-2) = q(q-1)^2 = |N_{G_q}^3((q, 1, 0)_0)|$, which implies that every $v \in N_{G_q}^3((q, 1, 0)_0)$ has exactly one neighbor in $N_{G_q}(R)$. It follows that $S := \bigcup_{z \in \mathbb{F}_q} N_{G_q}[(q, 1, z)_0] \cup N_{G_q}[R]$ is a perfect dominating set of G_q . Furthermore, by Lemma 2.3, $|S| = q^2 + q + q(3q-7) = 4q^2 - 6q$. Therefore a $(q-1)$ -regular graph of girth 8 can be obtained by deleting from G_q the indicated perfect dominating set S , see Figure 4. This graph has order $2q(q^2 - 2) - 2q(2q - 3) = 2q(q-1)^2$. ■

References

- [1] M. Abreu, M. Funk, D. Labbate, V. Napolitano, On (minimal) regular graphs of girth 6,

- Australas. J. Combin. 35 (2006) 119–132.
- [2] G. Araujo, C. Balbuena, and T. Héger, Finding small regular graphs of girths 6, 8 and 12 as subgraphs of cages, *Discrete Math.* 310(8) (2010) 1301–1306.
 - [3] E. Bannai and T. Ito, On finite Moore graphs, *J. Fac. Sci. Univ. Tokio, Sect. I A Math* 20 (1973) 191–208.
 - [4] C. Balbuena, Incidence matrices of projective planes and other bipartite graphs of few vertices, *Siam J. Discrete Math.* 22(4) (2008) 1351–1363.
 - [5] C. Balbuena, A construction of small regular bipartite graphs of girth 8, *Discrete Math. Theor. Comput. Sci.* 11(2) (2009) 33–46.
 - [6] L.M. Batten, *Combinatorics of finite geometries*, Cambridge University Press, Cambridge, UK, 1997.
 - [7] C.T. Benson, Minimal regular graphs of girth eight and twelve, *Canad. J. Math.* 18 (1966) 1091–1094.
 - [8] L. Beukemann and K. Metsch, Regular Graphs Constructed from the Classical Generalized Quadrangle $Q(4, q)$, *J. Combin. Designs* 19 (2010) 70–83.
 - [9] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, New York, 1996.
 - [10] N. Biggs, Construction for cubic graphs with large girth, *Electron. J. Combin.* 5 (1998) #A1.
 - [11] G. Brinkmann, B. D. McKay and C. Saager, The smallest cubic graphs of girth nine, *Combin. Prob. and Computing* 5 (1995) 1–13.
 - [12] P. Erdős and H. Sachs, Reguläre Graphen gegebener Taillenweite mit minimaler Knotenzahl, *Wiss. Z. Uni. Halle (Math. Nat.)*, 12 (1963) 251–257.
 - [13] G. Exoo, A Simple Method for Constructing Small Cubic Graphs of Girths 14, 15 and 16, *Electron. J. Combin.*, 3, 1996.
 - [14] G. Exoo and R. Jajcay, Dynamic cage survey, *Electron. J. Combin.* 15 (2008) #DS16.
 - [15] W. Feit and G. Higman, The non-existence of certain generalized polygons, *J. Algebra* 1 (1964) 114–131.
 - [16] A. Gács and T. Héger, On geometric constructions of (k, g) -graphs, *Contrib. to Discrete Math.* 3(1) (2008) 63–80.
 - [17] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer, NY 2000.

- [18] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of domination in graphs, Monogr. Textbooks Pure Appl. Math., 208, Dekker, New York, (1998).
- [19] D.A. Holton and J. Sheehan, The Petersen Graph, Chapter 6: Cages, Cambridge University (1993).
- [20] F. Lazebnik and V.A. Ustimenko, Explicit construction of graphs with an arbitrary large girth and of large size, Discrete Appl. Math. 60 (1995) 275–284.
- [21] F. Lazebnik, V.A. Ustimenko, and A.J. Woldar, New upper bounds on the order of cages, Electron. J. Combin. 4 (1997) # 2.
- [22] J. H. van Lint and R. M. Wilson, A course in Combinatorics, Cambridge University Press, UK 1994.
- [23] M. Meringer, Fast generation of regular graphs and construction of cages, J. Graph Theory 30 (1999) 137–146.
- [24] M. O’Keefe and P.K. Wong, The smallest graph of girth 6 and valency 7, J. Graph Theory 5(1) (1981) 79–85.
- [25] T. Pisanski, M. Boben, D. Marusic, A. Orbanic, A. Graovac, The 10-cages and derived configurations, Discrete Math. 275 (2004) 265–276.
- [26] W. T. Tutte, A family of cubical graphs. Proc. Cambridge Philos. Soc., (1947) 459–474.
- [27] P. K. Wong, Cages-a survey, J. Graph Theory 6 (1982) 1–22.